

# Efficient tilings of de Bruijn and Kautz graphs

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## Abstract

Kautz and de Bruijn graphs have a high degree of connectivity which makes them ideal candidates for massively parallel computer network topologies. In order to realize a practical computer architecture based on these graphs, it is useful to have a means of constructing a large-scale system from smaller, simpler modules. In this paper we consider the mathematical problem of uniformly tiling a de Bruijn or Kautz graph. This can be viewed as a generalization of the graph bisection problem. We focus on the problem of graph tilings by a set of identical subgraphs. Tiles should contain a maximal number of internal edges so as to minimize the number of edges connecting distinct tiles. We find necessary and sufficient conditions for the construction of tilings. We derive a simple lower bound on the number of edges which must leave each tile, and construct a class of tilings whose number of edges leaving each tile agrees asymptotically in form with the lower bound to within a constant factor. These tilings make possible the construction of large-scale computing systems based on de Bruijn and Kautz graph topologies.

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# 1 Introduction

The family of graphs known as de Bruijn graphs [1], Kautz graphs [2], and generalized de Bruijn and Kautz graphs [3, 4, 5, 6], have closely related mathematical structure. These graphs exhibit a high degree of connectivity, which makes them natural candidates for massively parallel computer network topologies [7]. In particular, for graphs of degree  $K$  and diameter  $N$ , de Bruijn and Kautz graphs achieve the largest possible asymptotic number of vertices ( $\sim K^N$ ) [8].

The efficient connectivity of de Bruijn and Kautz graphs makes it difficult to partition these graphs into collections of subgraphs in such a way as to minimize the number of edges connecting the subgraphs to one another. Precisely such a partition is desirable, however, in order to physically realize a computer network topology based on a graph of this type. The problem of constructing a computer based on a de Bruijn or Kautz graph from a set of isomorphic subgraphs (circuit boards) connected together by a minimal number of edges motivates the problem we consider in this paper of tiling de Bruijn and Kautz graphs with isomorphic subgraph tiles having a maximal number of internal edges.

In this paper, we define the tiling problem for de Bruijn and Kautz graphs. This problem can be thought of as a generalization of the graph bisection problem applied to these graphs, where instead of splitting the graph into two parts we wish to split the graph into a larger number of equal-size components. We derive a lower bound on the number of edges which must connect the tiles of a tiling. Based on the criterion that a tile decomposition should be scalable to construct graphs of arbitrary size, we give necessary and sufficient conditions for the construction of a uniform tiling. We construct a family of scalable tilings which asymptotically realize the lower bound on connecting edges to within a constant factor. We also provide examples of optimal tilings for small tiles. Finally, we show that not only de Bruijn and Kautz graphs but also their generalizations can be tiled in a similar way using the basic mathematical structure underlying de Bruijn graphs.

The structure of this paper is as follows. In section 2, we fix notation and state the fundamental problem addressed in the paper. In section 3, we derive a simple upper bound on the number of edges which can be incorporated into a tile of given size, equivalent to a lower bound on the number of connecting edges. In section 4, we prove necessary and sufficient conditions for the construction of a tiling of a de Bruijn or Kautz graph, and describe a general approach for the construction of tilings. Section 5 describes the generalization of the results of section 4 to generalized de Bruijn and Kautz graphs. Section 6 contains some examples of tilings constructed using the method of section 4. In section 7 we give an explicit construction of a family of tiles which asymptotically realize the lower bound of section 3 to within a constant factor, as the tile size becomes large. Section 8 contains a brief discussion of the application of the tiling approach developed in this paper to supercomputer architectures, including a mathematical argument for the optimality of degree 3 Kautz/de Bruijn graphs in certain contexts. Section 9 contains some brief concluding remarks.

A related formulation of this problem was given in [9], along with some explicit decompo-

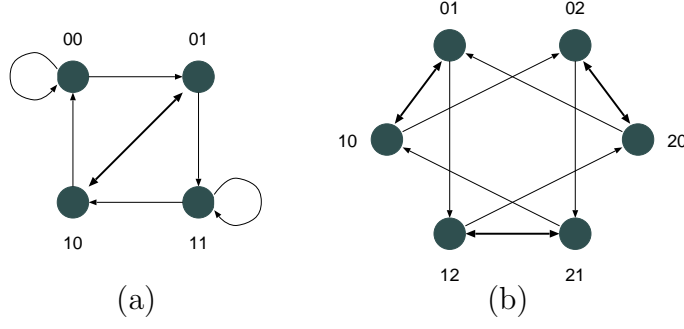


Figure 1: The de Bruijn graph  $\mathcal{B}_2^2$  (a) and Kautz graph  $\mathcal{K}_2^2$  (b) of degree 2 and diameter 2.

sitions of degree 2 de Bruijn graphs into isomorphic subgraph tiles similar to those described in this paper. We focus in this paper on the directed graph (digraph) form of de Bruijn and Kautz graphs. Similar considerations could be used for tiling the undirected forms of the graphs.

## 2 Definitions and problem statement

We begin by defining some notation.

First we define a (directed) de Bruijn graph. The de Bruijn graph  $\mathcal{B}_K^N$  of degree  $K$  and diameter  $N$  is a digraph with  $K^N$  vertices. The vertices can be labeled by strings of  $N$  integers base  $K$

$$c_1 c_2 \cdots c_N \in \mathbb{Z}_K^N, \quad (1)$$

so  $c_i \in \{0, 1, \dots, K-1\}$ . The de Bruijn graph  $\mathcal{B}_K^N$  has edges

$$(c_1 c_2 \cdots c_N, c_2 c_3 \cdots c_N c_{N+1}) \quad (2)$$

for all values of  $c_i, i \in \{1, \dots, N+1\}$ . There is thus a natural one-to-one map between edges of  $\mathcal{B}_K^N$  and vertices of  $\mathcal{B}_K^{N+1}$ . This leads naturally to an inductive construction of  $\mathcal{B}_K^{N+1}$  as a family of iterated line digraphs.

A (directed) Kautz graph  $\mathcal{K}_K^{N+1}$  of degree  $K$  and diameter  $N+1$  can be defined similarly as a digraph with  $(K+1)K^N$  vertices labeled by strings

$$s_0 s_1 \cdots s_N \in \mathbb{Z}_{K+1}^{N+1}, \quad s_i \neq s_{i+1}, \quad (3)$$

where the integers  $s_i$  are taken base  $K+1$ , but subject to the condition that adjacent integers must be distinct. The edges of  $\mathcal{K}_K^{N+1}$  are given in a similar fashion to (2) by

$$\{(s_0 s_1 \cdots s_N, s_1 s_2 \cdots s_N s_{N+1}) \mid s_i \in \mathbb{Z}_{K+1} \ s_i \neq s_{i+1}\}. \quad (4)$$

Some simple examples of de Bruijn and Kautz graphs for small values of the degree  $K$  and diameter are depicted in Figures 1-3

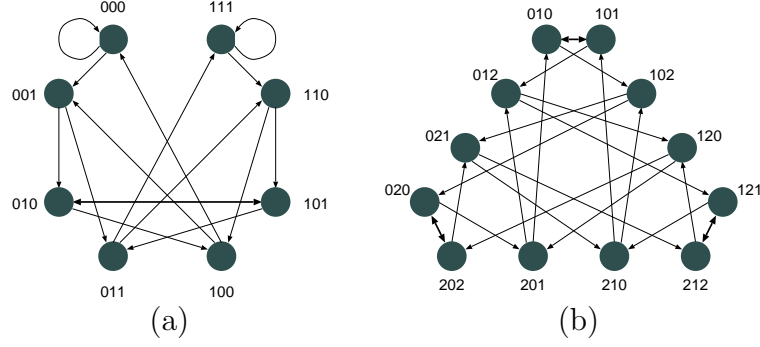


Figure 2: The de Bruijn graph  $\mathcal{B}_2^3$  (a) and Kautz graph  $\mathcal{K}_2^3$  (b) of degree 2 and diameter 3.

Kautz graphs and de Bruijn graphs both have uniform in-degree and out-degree  $K$ . Kautz graphs and de Bruijn graphs are closely related. Indeed, if we take a Kautz graph  $\mathcal{K}_K^{N+1}$  with vertices  $s_0 \cdots s_N$ , we can map the vertices to the vertices of the de Bruijn graph  $\mathcal{B}_K^N$  through

$$r : s_0 \cdots s_N \rightarrow c_1 \cdots c_N, \quad c_i = s_i - s_{i-1} - 1 \pmod{K+1}. \quad (5)$$

This map is a (many to one) graph homomorphism, since if  $(s_0 \cdots s_N, s_1 \cdots s_{N+1})$  is an edge of  $\mathcal{K}_K^{N+1}$  it follows immediately that the associated  $(c_1 \cdots c_N, c_2 \cdots c_{N+1})$  is an edge of  $\mathcal{B}_K^N$ . We can thus label every vertex in the Kautz graph by  $s_0 v$ , where  $s_0 \in \mathbb{Z}_{K+1}$  and  $v = c_1 c_2 \cdots c_N$  is a vertex in a de Bruijn graph  $\mathcal{B}_K^N$ . The edges of the Kautz graph in this notation are

$$(s_0 c_1 c_2 \cdots c_N, [s_0 + c_1 + 1] c_2 c_3 \cdots c_{N+1}) \quad (6)$$

where  $[s_0 + c_1 + 1]$  is taken modulo  $K+1$ . We will use this relationship between Kautz and de Bruijn graphs in the following section.

**Notation:** if a directed graph has an edge  $(u, v)$  we say that  $v$  is a *child* of  $u$  and  $u$  is a *parent* of  $v$ . We denote the set of children of  $u$  by  $C(u)$  and the set of parents of  $v$  by  $P(v)$ .

We are interested in finding tilings of de Bruijn and Kautz graphs with certain properties, where a tiling is defined as follows:

**Definition:** A *tiling* of a graph  $G$  is a one-to-one (on vertices) graph embedding  $t : I \times V_T \rightarrow V_G$  taking the product of an index set  $I = \{1, 2, \dots, |V_G|/|V_T|\}$  and the vertices  $V_T$  of a smaller graph  $T$  (the *tile*) into the vertices of the graph  $G$  such that if  $(x, y)$  is an edge of  $T$  then  $(t(i, x), t(i, y))$  is an edge of  $G$ ,  $\forall i$ .

We wish to find tilings of the de Bruijn and Kautz graphs  $\mathcal{B}_K^N$  and  $\mathcal{K}_K^{N+1}$  such that the number of edges of the tile  $T$  is maximized. This problem is motivated from the problem of designing a computer with a de Bruijn or Kautz topology, where we wish to construct the machine using a set of identical boards (the tiles), minimizing the number of wires connecting the boards by maximizing the number of on-board connections corresponding to edges of  $T$ .

**Problem 1** For fixed  $K, N$ , and tile size  $S = |V_T|$  such that  $S|K^N$ , find a tiling of  $\mathcal{B}_K^N$  which

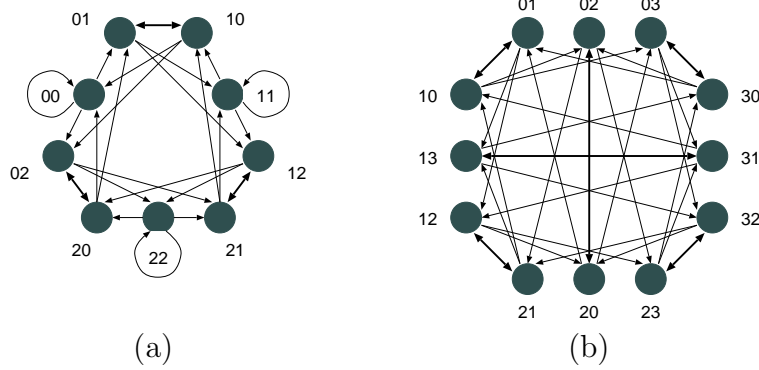


Figure 3: The de Bruijn graph  $\mathcal{B}_3^2$  (a) and Kautz graph  $\mathcal{K}_3^2$  (b) of degree 3 and diameter 2.

maximizes  $|E_T|$ , the number of edges in  $T$ .

This problem has previously been formulated in the case  $K = 2$ , and some example tilings described in [9].

**Problem 2** For fixed  $K, N$ , and tile size  $S|(K+1)K^N$ , find a tiling of  $\mathcal{K}_K^{N+1}$  which maximizes  $|E_T|$ .

**Definition:** a tiling is *scalable* if the associated tile  $T$  can be used to tile a de Bruijn or Kautz graph of any size  $K^N$  or  $(K+1)K^N \geq |V_T|$  (for fixed  $K$ ).

This property is useful practically because machines of different sizes can be built with the same modular component. The condition that the tiling be scalable also enables us to make more general statements about the set of allowed tilings.

**Problem 3:** For a fixed degree  $K$  and tile size  $S$ , find a tile  $T$  with  $S = |V_T|$  and maximum  $|E_T|$  which leads to a scalable tiling for either de Bruijn or Kautz graphs.

When  $K$  is prime, any tile must have size  $K^M$  for some  $M$ . For non-prime  $K$ , it may be possible to find tiles of other sizes giving scalable tilings. We will focus in this paper on tiles with sizes which are powers of  $K$ . Such tiles can be constructed for all  $K$ .

**Definition:** We define the maximum number of edges  $E_T$  which can be realized in any tile which has  $K^M$  vertices and which gives a scalable tiling for de Bruijn and Kautz graphs of degree  $K$  to be  $\hat{E}_{K,M}$ .

**Definition:** We define the minimum possible number of “broken” edges in any tile with  $K^M$  vertices to be  $\hat{B}_{K,M} = K^{M+1} - \hat{E}_{K,M}$ .

In this paper we construct a class of tilings which turn out to have the following useful property

**Definition:** a tiling has the *parallel routing* property if for each vertex  $x \in V_T$ , the children

of  $t(i, x)$  are mapped under  $t^{-1}$  to vertices in  $I \times V_T$  with the same set of  $V_T$  values (for each  $i$ ). In other words, defining the projection  $\Pi$  of a vertex in  $G$  to the tile  $T$  through

$$\Pi(t(i, x)) = x \quad \forall i \quad (7)$$

the parallel routing property states that

$$\Pi(C(t(i, x))) = \Pi(C(t(i', x))) \quad (8)$$

for any  $i, i'$  in the index set  $I$ .

This property is useful practically because it simplifies the wiring of a machine based on a tiling with this property.

### 3 Upper bound on internal tile edges

In this section we derive an upper bound on the number of edges in the tiles of a tiling, equivalent to a lower bound on the number of edges which must leave each tile. We assume here that tiles are of size  $K^M$ ; a similar bound can be found for tiles with sizes not of this form for composite  $K$  by the same analysis. The lower bound found here takes a simple asymptotic form as  $M \rightarrow \infty$  for fixed  $K$ . We focus on de Bruijn tilings, but analysis of Kautz tilings gives the same form for the bound.

**Theorem 1:** The number of edges achievable in a tile which leads to scalable tilings has an upper bound

$$\hat{E}_{K,M} \leq K^{M+1} - \max_N \frac{1}{N} (K^{M+1} - K^{2M+1-N}) \quad (9)$$

**Proof:**

We find this upper bound by considering a set of paths which leave the tile; we bound the number of these paths which cross any given edge, which gives us a lower bound on the number of edges which must leave the tile. This basic method, described in [11], is used in [10] to construct a lower bound on the edge bisection width of de Bruijn and Kautz graphs.

Consider a tiling of the de Bruijn graph  $\mathcal{B}_K^N$  by tiles of size  $K^M$ . For a given tile  $T$ , consider the  $K^M(K^N - K^M)$  paths of length  $N$  which go from a node in  $T$  to a node in the de Bruijn graph outside tile  $T$ . Each of these must cross at least one edge which begins at a node in  $T$  and ends at a node outside  $T$ . We refer to such an edge as an edge “leaving” the tile  $T$ . Each edge leaving the tile is traversed by at most  $NK^{N-1}$  of the paths under consideration (since the edge fixes  $N + 1$  of the  $2N$   $c_i$ ’s defining the path, but may appear at any of  $N$  places in the path). Thus, the number of edges leaving the tile ( $K^{N+1} - E_T$ ) is at least

$$K^{N+1} - E_T \geq \frac{K^M(K^N - K^M)}{NK^{N-1}} = \frac{1}{N} (K^{M+1} - K^{2M+1-N}) \quad (10)$$

for any choice of paths, and for any tile  $T$ . Thus we have shown

$$\hat{B}_{K,M} \geq \frac{1}{N} (K^{M+1} - K^{2M+1-N}) \quad (11)$$

for any  $N$ , which proves the theorem.

**Corollary** For fixed  $K$  and large  $M$  the lower bound on the number of edges leaving the tile goes asymptotically as

$$\hat{B}_{K,M} \geq \frac{1}{M} (K^{M+1}) (1 - \mathcal{O}(\log M/M)). \quad (12)$$

**Proof:**

By taking  $N = M + \lfloor \log_K M \rfloor$ , we have

$$\hat{B}_{K,M} \geq \frac{1}{N} (K^{M+1} - K^{2M+1-N}) \sim \frac{1}{M} (K^{M+1}) (1 - \mathcal{O}(\log M/M)). \quad (13)$$

In Section 7 we explicitly construct a class of tilings which realize this asymptotic bound to within a factor of 2.

The asymptotic lower bound we have found here for the number of broken edges is similar in form to that found in [10] for bisection of de Bruijn and Kautz graphs. Those authors found that the edge bisection width  $b_e$  satisfies the asymptotic inequalities

$$\frac{K^{N+1}}{2N} (1 - \mathcal{O}(1/K^{2N})) \leq b_e(K, N) \leq \frac{2K^{N+1}}{N} (1 + \mathcal{O}(N/K^{\sqrt{N}/2})). \quad (14)$$

Note that the asymptotic form of the lower bounds agrees for  $K = 2, N = M + 1$ , where the tiling and bisection problems are equivalent.

## 4 Structure of tiles

In this section we prove a necessary condition which a tile of size  $K^M$  must satisfy to lead to a scalable tiling. We then prove a related but slightly stronger sufficient condition for scalable tilings, by giving a general construction of a class of tilings based on any tile which satisfies the sufficient condition.

We begin with a few definitions

**Definition:** A *stratification* of a directed graph  $G$  is a map  $\sigma : V_G \rightarrow \mathbf{Z}$  from the vertices of  $G$  into the integers such that for every edge  $(u, v)$  in  $G$ , we have

$$\sigma(u) = \sigma(v) + 1. \quad (15)$$

We refer to  $\sigma(u)$  as the *level* of the vertex  $u$  in the stratified graph  $G$ . By convention, we take the smallest level in a stratification to be 0 by shifting all levels equally. We refer to a

graph which admits a stratification as *stratifiable*.

**Definition:** A *loop* is a directed graph which is topologically equivalent to a circle.

Note that the edges in a loop need not have compatible orientations. It is straightforward to show that a loop is stratifiable if and only if it contains an equal number of edges of each orientation. For example, the loop defined by the graph containing 3 vertices  $u, v, w$  and edges  $\{(u, v), (v, w), (u, w)\}$  is unstratifiable since a stratification would give  $\sigma(u) = \sigma(v) + 1 = \sigma(w) + 2$  from the first two edges and  $\sigma(u) = \sigma(w) + 1$  from the third edge.

**Lemma 1:** A graph  $G$  is stratifiable if and only if it contains no unstratifiable loops as subgraphs.

**Proof:** If  $G$  contains an unstratifiable loop  $L$  then  $G$  is clearly not itself stratifiable since any stratification of  $G$  would provide a stratification of  $L$ . If  $G$  contains no unstratifiable loops, then we can explicitly construct a stratification of  $G$  by choosing a single reference vertex in each connected component of  $G$  and assigning the level of each other vertex  $w$  in  $G$  by implementing the rule (15) on each of the edges needed to reach  $w$  on any path from a reference vertex, allowing the path to traverse edges in either direction. In the absence of unstratifiable loops, this assignment is path independent and provides a stratification of  $G$ .

We can now prove the following

**Theorem 2:** A necessary condition for a tile  $T$  of size  $K^M$  to give scalable tilings of  $\mathcal{B}_K^N$  and  $\mathcal{K}_K^{N+1}$  for all  $N \geq M$  is that  $T$  must be stratifiable.

**Proof:** We begin by assuming that  $T$  is unstratifiable and therefore contains an unstratifiable loop. We will show that this assumption leads to the conclusion that  $T$  cannot be used to construct scalable tilings, thus proving the theorem by contradiction.

Assuming that a given tile  $T$  contains an unstratifiable loop  $L$  of length  $n$ , we write  $L$  in terms of the sequence of vertices in  $T$  it traverses.

$$\begin{aligned} L &= [u_1, u_2, \dots, u_n, u_{n+1} = u_1] \\ u_{i+1} &\in C(u_i), i \in l_+ \\ u_{i+1} &\in P(u_i), i \in l_- \end{aligned} \tag{16}$$

where  $l_+, l_-$  denote the sets of indices of vertices associated with edges  $(u_i, u_{i+1})$  which are traversed in a forward/backward direction in the loop  $L$ , so that  $l_- \cap l_+ = \{\}$ ,  $l_- \cup l_+ = \{1, 2, \dots, n\}$ . Since  $L$  is unstratifiable, we must have

$$|l_+| \neq |l_-|. \tag{17}$$

Now, imagine that the tile  $T$  gives a scalable tiling of de Bruijn graphs  $\mathcal{B}_K^N$  for  $N \geq M$  but nonetheless contains the unstratifiable loop  $L$  given in (16). In a tiling of the graph  $\mathcal{B}_K^N$ , there will be  $K^{N-M}$  nodes  $v$  which project to each node  $u \in V_T$  through  $u = \Pi(v)$ .



In particular, there will be  $K^{N-M}$  nodes  $v^{(i)} = t(i, u_1)$  with  $\Pi(v^{(i)}) = u_1$ . Choose any such node  $v_1 = c_1 \cdots c_N$  which can be written as  $t(i_0, u_1) = v_1$  for some fixed  $i_0$ . Then we can define a loop in  $\mathcal{B}_K^N$  through

$$v_i = t(i_0, u_i) \quad (18)$$

whose vertices must satisfy

$$v_{i+1} \in C(v_i), \text{ if } i \in l_+, \quad (19)$$

$$v_{i+1} \in P(v_i), \text{ if } i \in l_- \quad (20)$$

Since  $|l_+| \neq |l_-|$ , when we follow the loop around, the string of characters defining  $v_1$  has been shifted to the left  $|l_+| - |l_-| \neq 0$  times. But we have returned to  $v_1$  after going around the full loop; this implies that the bulk of the word  $v_1 = c_1 \cdots c_N$  is invariant under shifting left by  $|l_+| - |l_-|$ . Specifically,

$$c_i = c_{i+\Delta}, \text{ where } \Delta = |l_+| - |l_-|, \quad i > n, \quad i \leq N - n - \Delta \quad (21)$$

But only  $K^{2n+\Delta}$  nodes have this property (such nodes are completely defined by giving  $c_1 \cdots c_n, c_{N-n-\Delta+1} \cdots c_N$ ), so we cannot have  $K^{N-M}$  nodes with  $\Pi(v) = u_1$  when  $N > M + 2n + \Delta$ . Thus, the assumption of an unstratifiable loop is incompatible with the assumption of a scalable tiling for  $T$ , so we conclude that if  $T$  can be used to construct a scalable tiling it cannot contain an unstratifiable loop, and therefore must be stratifiable. The theorem is thus proven.

The condition of stratifiability is not by itself sufficient to guarantee that a tile gives rise to a scalable tiling. In some cases, stratifiable loops can form obstructions to the construction of a tiling. We now will proceed to prove that scalable tilings can be constructed on stratifiable tiles with certain types of loops. First, however, we need a few more definitions and a lemma.

**Lemma 2:** A tile of size  $S = K^M$  giving scalable tilings of  $\mathcal{B}_K^N$  and  $\mathcal{K}_K^{N+1}$  for  $N \geq M$  must be a stratifiable subgraph of  $\mathcal{B}_K^M$

**Proof:** The tile must be stratifiable by Theorem 2. It must be a subgraph of  $\mathcal{B}_K^M$  since it must tile  $\mathcal{B}_K^N$  with  $N = M$ . Note that since  $S = K^M$ , the subgraph contains all vertices of  $\mathcal{B}_K^M$ , but will only contain a subset of the edges.

We can thus associate with any scalable tiling of a de Bruijn or Kautz graph  $G = \mathcal{B}_K^N$  or  $\mathcal{K}_K^{N+1}$  a map  $\Pi : V_G \rightarrow V_{\mathcal{B}_K^M}$ . For the tilings which we will construct, this map extends to a complete graph homomorphism  $\Pi : G \rightarrow \mathcal{B}_K^M$ . We define several properties for maps of this type

**Definition:** A map  $\Pi : G \rightarrow \mathcal{B}_K^M$  has the *parent distribution* property when

$$\Pi(P(u)) = P(\Pi(u)), \quad \forall u \in G \quad (22)$$

**Definition:** A map  $\Pi : G \rightarrow \mathcal{B}_K^M$  has the *child distribution* property when

$$\Pi(C(u)) = C(\Pi(u)), \quad \forall u \in G \quad (23)$$

Note in particular that if  $\Pi : G \rightarrow \mathcal{B}_K^M$  has both the parent and child distribution properties, then for any vertex  $u \in G$  with  $\Pi(u) = d_1 \cdots d_M \in \mathcal{B}_K^M$ , there is a unique parent  $p_{u,x} \in P(u)$  of  $u$  with  $\Pi(p_{u,x}) = xd_1 \cdots d_{M-1}$  and a unique child  $c_{u,x} \in C(u)$  of  $u$  with  $\Pi(c_{u,x}) = d_2 \cdots d_M x$  for each  $x \in \mathbf{Z}_K$ .

We now prove by example that there exist maps  $\Pi : G \rightarrow \mathcal{B}_K^M$  with  $G = \mathcal{B}_K^N$  or  $\mathcal{K}_K^{N+1}$  which have both the parent and child distribution properties, for any  $N \geq M$

For a de Bruijn graph  $\mathcal{B}_K^N$ , we begin by defining the  $k$ th *discrete differentials*. For each vertex  $v = c_1 \cdots c_N$  we define

$$\begin{aligned} d_i^0(v) &= c_i \\ d_i^k(v) &= c_{i+k} - c_i \pmod{K}, \quad i = 1, \dots, N-k, \quad k > 0. \end{aligned} \quad (24)$$

Let us fix  $N \geq M$  and take  $k = N - M$ . Then the discrete differentials define a map

$$d : c_1 \cdots c_N \rightarrow d_1 \cdots d_{N-k}, \quad d_i = d_i^k(c_1 \cdots c_N) \quad (25)$$

which is a graph homomorphism

$$d : \mathcal{B}_K^N \rightarrow \mathcal{B}_K^M \quad (26)$$

since if  $(c_1 \cdots c_N, c_2 \cdots c_{N+1})$  is an edge of  $\mathcal{B}_K^N$  then  $(d_1 \cdots d_{N-k}, d_2 \cdots d_{N-k+1})$  is an edge of  $\mathcal{B}_K^{N-k}$ . It is straightforward to check that this graph homomorphism has both the child and parent distribution properties. Note that using this homomorphism we can relabel the vertices of the graph  $G = \mathcal{B}_K^N$  using the first  $k$   $c$ 's and the  $d$ 's

$$V_G = \{c_1 \cdots c_k d_1 \cdots d_{N-k}\}. \quad (27)$$

Combining the discrete differentials on the de Bruijn graph  $\mathcal{B}_K^N$  with the homomorphism (5) gives us a map

$$\tilde{d} : \mathcal{K}_K^{N+1} \rightarrow \mathcal{B}_K^M \quad (28)$$

defined through  $\tilde{d}(u) = d(r(u))$ , with  $r$  defined as in (5). Again, it is easy to verify that this graph homomorphism has the child and parent distribution properties. Using this homomorphism, we can label the nodes in a Kautz graph  $H = \mathcal{K}_K^{N+1}$  through

$$V_H = \{s_0 c_1 \cdots c_k d_1 \cdots d_{N-k} | s_0 \in \mathbb{Z}_{K+1}, c_1, \dots, d_{N-k} \in \mathbb{Z}_K\}. \quad (29)$$

**Definition:** We define the *height* of a stratifiable loop  $L$  to be the difference  $\sigma_{\max} - \sigma_{\min}$  between the largest and smallest levels in a loop.

We can now prove the following

**Theorem 3** Given a stratified tile  $T$  of size  $K^M$  which is a subgraph of  $\mathcal{B}_K^M$  (containing all vertices but not all edges of  $\mathcal{B}_K^M$ ), tilings of  $G = \mathcal{B}_K^N$  and  $\mathcal{K}_K^{N+1}$  can be constructed for all  $N \geq M$ , as long as  $T$  contains no loops of height  $> M$ . Such a tiling can be constructed explicitly from any map  $\Pi : G \rightarrow \mathcal{B}_K^M$  with the child and parent distribution properties.

**Proof:** We showed above by explicit construction that a map  $\Pi : G \rightarrow \mathcal{B}_K^M$  with the child and parent distribution properties exists for any  $M$ , for  $G$  a de Bruijn or Kautz graph of degree  $K$  with  $N \geq M$ . Let us take any particular such  $\Pi$ . From  $\Pi$  we can define the tiling constructively as follows: choose a vertex  $x_0 = d_1 \cdots d_M$  of the tile  $T$ . Associated with this vertex of  $T$  there is a set of vertices in our de Bruijn or Kautz graph  $G$  which map to  $x_0$  under  $\Pi$ . We can arbitrarily associate this set of vertices with the elements  $i$  in the index set  $I$  to define  $t(i, x_0)$  for all  $i$ . For example, using the discrete differential map described above, given a specific  $x_0 = d_1 \cdots d_M$  we can define the index set by the leading indices before the  $d$ 's on this set of vertices in the notation of (27, 29) (*i.e.*,  $c_1 \cdots c_k$  for de Bruijn,  $s_0 c_1 \cdots c_k$  for Kautz).

We have now defined  $t(i, x_0)$  for all  $i$  and for a specific  $x_0$  in  $V_T$ . Now, if there are any edges in  $T$  containing  $x_0$  (*i.e.* directed edges  $(x_0, x')$  or  $(x', x_0)$  beginning or ending on  $x_0$ ), we can extend the definition. Say  $T$  contains the edge  $(x_0, x_1)$ . For each  $i \in I$ , using the child distribution property, there is a unique vertex  $v_i$  in  $G$  which is a child of  $t(i, x_0)$  and which has  $\Pi(v_i) = x_1$ . We can thus define  $t(i, x_1) = v_i$  for each  $i$ . These vertices  $v_i, v_j$  are distinct for  $i \neq j$ , by the parent distribution property—if  $v_i = v_j$  then  $t(i, x_0)$  and  $t(j, x_0)$  would both be parents of the same vertex with the same value of  $\Pi$ , which is impossible given the parent distribution property. Instead of extending along an edge where  $x_0$  is the parent, we could equivalently extend along an edge where  $x_0$  is the child, using an equivalent argument where the roles of parents and children are exchanged. We continue in this way for further edges containing either  $x_0$  or  $x_1$ . Each time we include a new edge we define the map  $t$  for a new value of  $x$ . If the graph  $T$  is disconnected, we run out of edges before the map  $t$  is completely constructed. In this case we choose a new vertex  $x'_0$  where  $t$  is not yet defined and proceed as above with  $x'_0$  in place of  $x_0$ . In this fashion, we can construct the complete tiling.

It remains to be shown that the construction of the tiling we have just presented is well-defined in that any loops existing in the tile do not lead to incompatible definitions for the tiling by following different paths to extend the definition of the tiling. This is straightforward to demonstrate for any loop whose height is less or equal to  $M$ . Given a loop  $L \subset T$  of height  $h \leq M$  we need to show that the loop can be consistently lifted to a loop in the graph  $G$  by following the edges and using the child and parent distribution properties

as in the above construction. Let us start at a vertex  $x_1 \in L$  such that  $\sigma(y) \geq \sigma(x_1) \forall y \in L$ . We lift  $x_1 = d_1 \cdots d_M$  to a vertex  $u_1 \in G$  with  $\Pi(u_1) = x_1$ . Labeling the vertices of the loop in order as  $[x_1, \dots, x_s, x_{s+1} = x_1]$  where  $s$  is the length of the loop we can use the child and parent distribution properties to lift each  $x_i$  to a vertex  $u_i \in G$  as above, following the edges of the loop one after another. We need to show that when we return to  $x_1$  the final value of  $u_{s+1}$  is the same as  $u_1$ . Assume that  $G = \mathcal{B}_K^N$ —a similar argument proceeds for Kautz graphs. We can write  $u_1 = c_1 \cdots c_N$ . From the form of the de Bruijn graph edges, and since  $x_1$  has the lowest level of any vertex in the loop  $L$ , we see that all vertices  $u_i$  contain the coordinates  $c_1 \cdots c_{N-h}$ , shifted to the right by  $\sigma(x_i) - \sigma(x_0)$  places. But then,  $u_{s+1} = c_1 \cdots c_{N-h} \tilde{c}_{N-h+1} \cdots \tilde{c}_N$ . Since by the child distribution property there is a unique  $M$ th child of  $z_1 \cdots z_M c_1 \cdots c_{N-M}$  with  $\Pi = x_1$ , we must have  $u_{s+1} = u_1$  when  $h \leq M$ , so the loop of height  $h \leq M$  does not obstruct the construction of the tiling. This completes the proof of the theorem.

**Corollary:** The tilings constructed by the method of Theorem 3 have the parallel routing property.

**Proof:** This follows from the use of  $\Pi$  with parent and child distribution properties, since

$$\Pi(C(t(i, x))) = C(\Pi(t(i, x))) = C(x) \quad (30)$$

is independent of  $i$  so (8) holds for all  $i, i', x$ . The parallel routing property can also be seen to follow immediately from the fact that  $\Pi$  is a graph homomorphism.

We have now provided results which make it possible to systematically search for optimal solutions to Problem 3. We know that for tile size  $S = K^M$ , scalable tilings are given by stratifiable subgraphs of  $\mathcal{B}_K^M$ . We can thus consider the set of stratifiable subgraphs of  $\mathcal{B}_K^M$ , ordered by their number of edges. We know that all stratifiable subgraphs with no loops of height  $h > M$  can give scalable tilings. Thus, if a stratifiable subgraph of  $\mathcal{B}_K^M$  can be realized with the maximal number of edges and  $h \leq M$ , this provides a solution to Problem 3. In some cases, the stratifiable subgraph of  $\mathcal{B}_K^M$  with the maximal number of edges may have a loop of height  $h > M$ . In such a case, a further check is needed to verify that such a subgraph gives a scalable tiling; generally this will not be possible, but it may be possible in special cases. In any case, finding a subgraph of  $\mathcal{B}_K^M$  with the maximal number of edges subject to the condition that there are no loops of height  $h > M$  will give rise to a good set of scalable tilings.

We conclude this section with some further comments.

First, we note that the discrete differential construction above is not the only way to realize a map  $\Pi : G \rightarrow \mathcal{B}_K^M$  with the parent and child distribution properties. Indeed, consider any function

$$f : \mathbf{Z}_K \times \mathbf{Z}_K \rightarrow \mathbf{Z}_K \quad (31)$$

with the following properties:

- for each  $c \in \mathbf{Z}_K$ ,  $f(c, \cdot)$  gives a one-to-one map from  $\mathbf{Z}_K \rightarrow \mathbf{Z}_K$  (i.e., for fixed  $c$ ,  $f(c, c')$  takes different values for each  $c'$ )
- for each  $c' \in \mathbf{Z}_K$ ,  $f(\cdot, c')$  gives a one-to-one map from  $\mathbf{Z}_K \rightarrow \mathbf{Z}_K$  (i.e., for fixed  $c'$ ,  $f(c, c')$  takes different values for each  $c$ )

If we replace  $d_i^k = c_{i+k} - c_i \pmod K$  in (24) with any other function  $d_i^k = f(c_{i+k}, c_i)$  with these properties, we see that the associated map  $\Pi : G \rightarrow \mathcal{B}_K^M$  still has the child and parent distribution properties when  $G$  is either a de Bruijn or Kautz graph. Using such a more general map in the construction of Theorem 3 provides a more general class of constructions of scalable tilings with the parallel routing property.

Note however that not every graph homomorphism of the form  $\Pi : G \rightarrow \mathcal{B}_K^M$  has the parent and child distribution properties. For example, while the map  $\mathcal{B}_K^N \rightarrow \mathcal{B}_K^j$  defined by taking a contiguous subset  $c_a c_{a+1} \cdots c_{a+j-1}$  of characters in the string  $c_1 \cdots c_N$  is a homomorphism, it does not have these properties, and indeed cannot be used to define a tiling. To see this clearly in a specific simple case, consider the case  $K = 3, N = 3$ . We cannot tile the 27-node de Bruijn graph  $\mathcal{B}_3^3$  using tiles with vertices addressed by  $c_2 c_3$ . For example, the 3 parents of the vertex  $u = 001$  have coordinates  $c_0 c_1 c_2 = 000, 100, 200$ . These vertices all have the same value of  $c_2 c_3 = 00$ , so the graph homomorphism taking  $c_0 c_1 c_2 \rightarrow c_1 c_2$  does not have the parent distribution property. Assume now for example that our tile includes the edge  $(00, 01)$ . The vertex  $101$ , which should be associated with the vertex  $01$  on the tile has no parent with  $c_2 c_3 = 00$ , so the edge  $(00, 01)$  cannot be contained in the tile. Similar problems arise with any other edge, so we cannot tile  $\mathcal{B}_3^3$  with tiles having vertices addressed by  $c_2 c_3$  and more than 0 edges.

## 5 Tilings of generalized de Bruijn and Kautz graphs

In this section we describe tilings of generalized de Bruijn and Kautz graphs. So far, we have discussed only de Bruijn and Kautz graphs of sizes  $K^N$  and  $(K+1)K^N$  respectively. Generalizations of the de Bruijn and Kautz graphs to other sizes are described in [3, 4, 5, 6]. We show here that the construction of the previous section gives tilings by tiles of size  $S = K^M$  for generalized de Bruijn and Kautz graphs with  $V = nS$  vertices for any integer  $n$ .

### 5.1 Generalized de Bruijn graphs

A generalized de Bruijn graph of degree  $K$  having  $V$  vertices can be defined by taking the directed graph on vertices  $0 \leq i < V$  with edges

$$(i, (Ki + m) \bmod V), \quad \forall m, 0 \leq m < K. \quad (32)$$

If  $V = K^N$ , it is easy to see that this definition agrees with the one in (2) by simply taking  $c_1 \cdots c_N$  to be the base  $K$  representation of the integer  $i$  for each vertex  $i$ . Multiplication by

$K$  simply shifts the base  $K$  representation left by one digit, adding  $m$  shifts in an arbitrary new digit  $c_{N+1}$  and modding by  $V = K^N$  truncates to  $N$  digits base  $K$ . More generally, we can write any  $V = nK^M$  in the form

$$V = FK^N, \quad (33)$$

where  $M \geq N$  and  $F$  and  $K$  are relatively prime, *i.e.*  $(F, K) = 1$ . We can then represent any vertex  $i < V$  in mixed-base form as

$$fc_1 \cdots c_N, \quad c_i < K, f < F. \quad (34)$$

As before, the transformation rule associated with edges (32) shifts the  $c$ 's left. This transformation rule takes  $fc_1 \cdots c_N \rightarrow f'c_2 \cdots c_{N+1}$  where

$$f' = (Kf + c_1) \bmod F. \quad (35)$$

Since  $K$  and  $F$  are relatively prime, multiplication by  $K$  is invertible mod  $F$ , so that for any fixed  $c_1$  the map defined in (35) from  $f \rightarrow f'$  is a one-to-one map. In particular, given  $c_1, f'$  there exists a unique  $f < F$  such that (35) is satisfied. This demonstrates that the map from the generalized de Bruijn graph with  $V$  vertices to the regular de Bruijn with  $K^N$  vertices given by dropping the first digit  $f$  in (34) has the parent distribution property (22). This map also clearly has the child distribution property (23), which follows from the fact that the children of any node with mixed-base representation  $fc_1 \cdots c_N$  are just  $f'c_2 \cdots c_N m$  with  $f'$  given by (35) and  $0 \leq m < K$ . Since this map has both the parent and child distribution properties, so does any map to a smaller de Bruijn formed by the composition of this map with a map  $\Pi' : \mathcal{B}_K^N \rightarrow \mathcal{B}_K^M$  also having the parent and child distribution properties. This shows that the tiles which can be used to construct tilings of standard de Bruijn and Kautz graphs in Theorem 3 of the previous section can also be used to tile generalized de Bruijn graphs. We give some examples of such tilings in Section 6

## 5.2 Generalized Kautz graphs

Similar to the generalized de Bruijn graphs defined through (32), a Kautz graph of degree  $K$  having  $V$  vertices can be defined by taking the directed graph on vertices  $0 \leq i < V$  with edges

$$(i, (-1 - Ki - m) \bmod K), \quad \forall m, 0 \leq m < V. \quad (36)$$

To relate this to the standard Kautz graph in the case  $V = (K + 1)K^N$ , we again introduce a mixed mode representation for each vertex  $i$ , writing  $V = FK^N$  with  $(K, F) = 1$ . We denote the digits in this mixed mode representation by

$$f\bar{c}_1 c_2 \bar{c}_3 c_4 \cdots \frac{c_N}{\bar{c}_N}, \quad c_i, \bar{c}_j < K, f < F \quad (37)$$

where the last digit is  $\bar{c}_N$  if  $N$  is odd, and  $c_N$  if  $N$  is even. The barred digits are defined to be  $\bar{c}_i = K - 1 - c_i$ , giving a shift register representation of the vertices. From (36) we see that the graph edges are given by

$$(f\bar{c}_1c_2\cdots\begin{smallmatrix}c_N\\ \bar{c}_N\end{smallmatrix}, f'\bar{c}_2c_3\cdots\begin{smallmatrix}\bar{c}_N\\ c_N\end{smallmatrix}m) \quad (38)$$

where

$$f' = (-1 - Kf - \bar{c}_1) \bmod F \quad (39)$$

$$= (-Kf + c_1 - K) \bmod F. \quad (40)$$

In particular, if we choose  $F = K + 1$ , we have  $K \equiv -1 \pmod{F}$ , so  $f' = (f + 1 + c_1) \bmod F$ , which is precisely the transformation rule on edges in (6) if we identify  $f = s_0$ . Thus, when  $F = K + 1$  we have the standard Kautz graph, while for other  $F$  we have a generalization.

As in the previous subsection, the invertibility of multiplication by  $K \bmod F$  guarantees that the map to a  $K^N$  node de Bruijn given by dropping the first digit  $f$  in the mixed mode representation (37) has the parent and child distribution properties, and therefore the generalized Kautz graphs can also be tiled using the methods of the previous section. Examples are given in the following section.

## 6 Examples

In this section we give some explicit examples of tilings constructed using the discrete differential method of Theorem 3.

Let us first consider tilings of de Bruijn graphs with  $K = 2$  by tiles of size 4. The de Bruijn graph  $\mathcal{B}_2^2$  is shown in Figure 1. Up to graph isomorphisms there are 4 kinds of loops in this graph:

$$[00 \rightarrow 00], \quad [00 \rightarrow 01 \rightarrow 10 \rightarrow 00], \quad [00 \rightarrow 01 \leftarrow 10 \rightarrow 00], \quad [00 \rightarrow 01 \rightarrow 11 \rightarrow 10 \rightarrow 00], \quad (41)$$

where the second and third loops differ only in the orientation of the link connecting 01 and 10. All these loops are non-stratifiable. Thus, there are no stratifiable tiles given by subgraphs of  $\mathcal{B}_2^2$  with more edges than the tile with 3 edges:

$$E_T = \{(00, 01), (01, 11), (11, 10)\}. \quad (42)$$

This tile therefore gives a scalable tiling which solves Problem 3 for  $K = 2, S = 4 = 2^2$ . As an example of a tiling using this tile, consider the tiling of  $\mathcal{B}_2^3$  using the map  $\Pi$  described using discrete differentials (26). This tiling is depicted in Figure 4. In this figure the numbers in brackets are the addresses  $c_1d_1d_2$  for each node as in (27). The colored/bold links are those realized on a tile  $T$  described by (42).

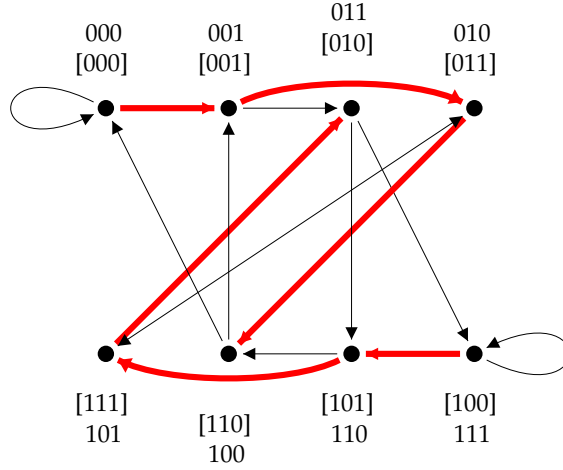


Figure 4: Optimal scalable tiling of the de Bruijn graph  $\mathcal{B}_2^3$  by tiles of size 4

Let us now consider simple examples of a Kautz graph tiling with  $K = 2$ . The optimal size 4 ( $M = 2$ ) tile for  $K = 2$  again has no loops and has only 3 edges. We can, for example choose the stratified subgraph of  $\mathcal{B}_2^2$  containing edges  $E_T = \{(00, 01), (10, 01), (01, 11)\}$ , with  $\sigma(00) = \sigma(10) = 2, \sigma(01) = 1, \sigma(11) = 0$ . We can use the discrete differential map (28)  $\tilde{d} : \mathcal{K}_2^3 \rightarrow \mathcal{B}_2^2$  to label the vertices of the Kautz graph  $\mathcal{K}_2^3$  by  $s_0 d_1 d_2$ . The tiling associated with this map is depicted in Figure 5, where nodes are labeled by  $s_0 s_1 s_2$  (the nodes are, however, ordered according to the index  $i$  from (36)). In this figure, again colored/bold links are realized on a tile  $T$ . The direction of each link is indicated by using dotted lines on the outgoing part of the link and solid lines on the incoming part of the link.

For  $K = 2$ , up to relabeling there is only one function of the form (31) with the desired properties, which is the one used in the definition (24) of discrete differentials. For  $K = 3$ , it is easy to check that up to relabeling of the integers, which is a symmetry of both the de Bruijn and Kautz graphs, there are precisely two distinct functions with the desired properties:

$$f_1(c, c') = c - c' \pmod{K} \quad (43)$$

and

$$f_2(c, c') = c + c' \pmod{K} \quad (44)$$

Either of these functions can be used to construct tilings with  $K = 3$ .

Now, for  $K = 3$ , consider the tile with  $M = 1$ . The de Bruijn graph here is just the 3 vertices 0, 1, 2 with edges going from each vertex to each other vertex. The largest subgraph without directed loops has two edges, such as  $0 \rightarrow 1 \rightarrow 2$ . This defines the best tile for a scalable tiling. Now consider  $M = 2$ . A systematic analysis of possible stratifiable subgraphs of  $\mathcal{B}_3^2$  shows that the maximum number of edges compatible with stratifiability is 11, which can be realized for example by the tile depicted in Figure 6. This tile has 4 loops of length



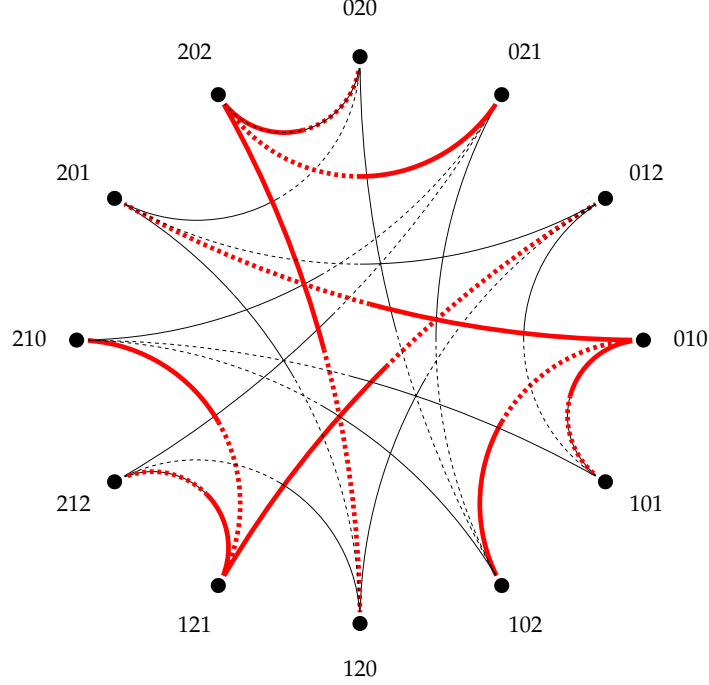


Figure 5: Optimal scalable tiling of the Kautz graph  $\mathcal{K}_2^3$  by 3 tiles of size 4. Direction of link goes from dotted end to solid end.

4 (of which 3 are homotopically independent), all of which are of height 1. Tilings of the generalized de Bruijn and Kautz graphs with 18 nodes by two copies of this tile are shown in Figures 7 and 8. In these figures the node numbers are the numbers  $i$  used in the description of generalized de Bruijn and Kautz graphs in Section 5.

We have carried out a systematic analysis of all tiles for small values of  $K, N$ . We have computed the optimal tiles by performing a complete search over all possibilities in all cases where  $K^N \leq 16$ . Our results are tabulated here.

$K$	$N$	$K^N$	$\max  E_T $
2	2	4	3
2	3	8	8
3	2	9	11
2	4	16	19
4	2	16	27

Using a simple greedy-first algorithm we have found some good, but not necessarily optimal tiles for larger values of  $K^N$ ; as examples, for  $K = N = 3$  there is a tile with 44 internal edges, and with  $K = 3, N = 4$  there is a tile with 150 internal edges. In the next section we discuss tilings for arbitrarily large values of  $M$

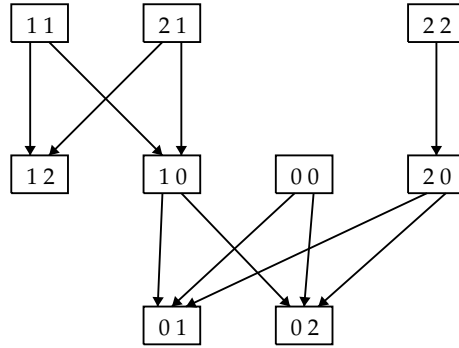


Figure 6: Optimal tile for  $K = 3$ ,  $M = 2$ . Tile has 11 internal edges (shown) and 7 external edges (not shown)

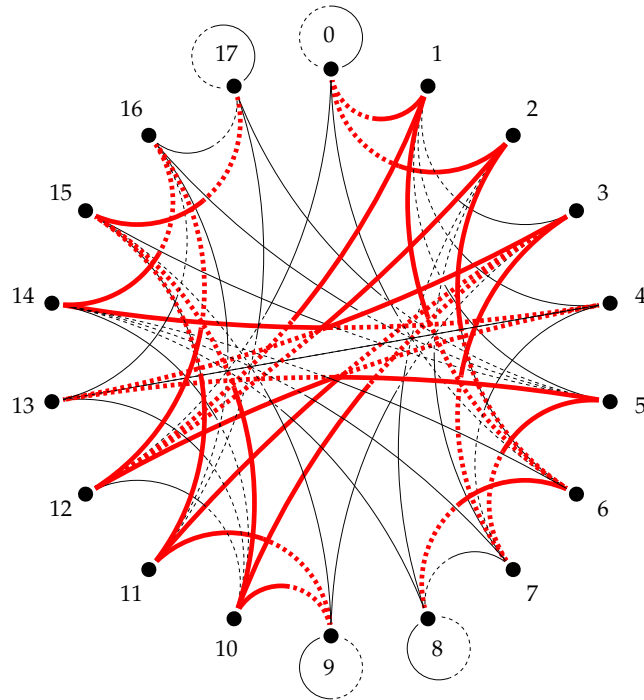


Figure 7: Optimal scalable tiling of the generalized de Bruijn graph with  $K = 3$ ,  $V = 18$  by tiles of size 9

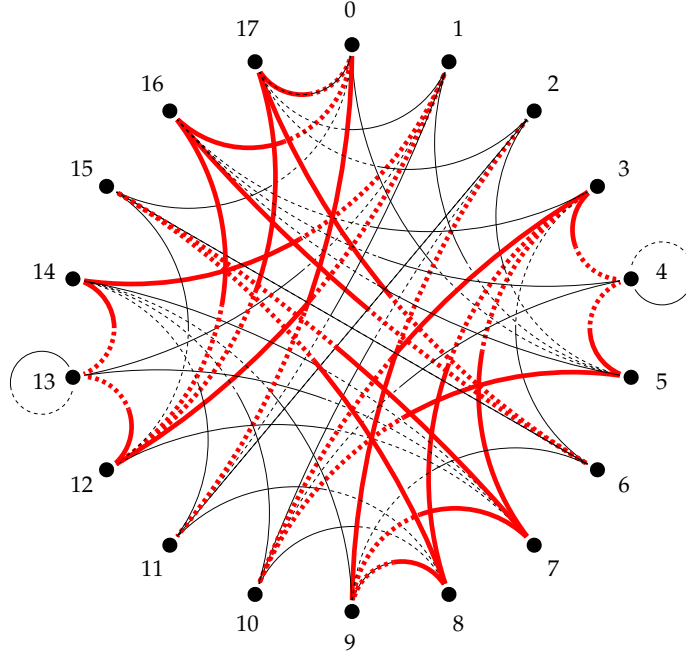


Figure 8: Optimal scalable tiling of the generalized Kautz graph with  $K = 3, V = 18$  by tiles of size 9

## 7 Asymptotically efficient tilings

In this section we provide an explicit construction of an efficient class of tilings. This class of tilings realizes the asymptotic bound (12) on edges leaving the tile to within a factor of 2. As we discuss below, we believe that this class of tilings may in fact be asymptotically optimal, and that the theoretical bound may be improvable by a factor of 2. We cannot, however, demonstrate this conclusively at this time. Note that for the graph bisection problem there is a similar gap (in that case by a factor of 4) between upper and lower bounds on edge bisection bandwidth (14) for general  $K, N$  [10].

### 7.1 Local score-based tiles

We will now define some tiles explicitly by choosing a stratification of the nodes in a de Bruijn graph and applying Theorem 3. The basic idea we will use to construct efficient tilings is the identification of local patterns in the  $dd$  (discrete differential) address space to choose the levels in the stratification of the nodes of the tile of size  $K^M$ .

For each node of the de Bruijn graph  $\mathcal{B}_K^M$  labeled by  $d_1 \cdots d_M$ , we assign to each position in the node address a score between 0 and 1 according to how well the node address in the vicinity of  $d_i$  matches some desired pattern. We will use the convention that a smaller score indicates a closer match to the desired pattern. The idea is that if this score is very low at

some particular position  $i$  for a node  $x = d_1 \cdots d_M$ , then the score will generally be low for position  $i - 1$  in the node addresses of all the children of  $x$ . Thus, if we choose the level of the node  $x$  to be  $\sigma(x) = i$  where  $i$  is the position of lowest score, then generically children  $y$  of  $x$  will have lowest score at position  $i - 1$ , and will be assigned level  $\sigma(y) = i - 1$ , so that the edge  $(x, y)$  can be included in the tile.

There are many ways in which we can define such a local pattern-based score. To give a concrete example we define such a score in a way which leads to efficient tilings.

**Definition:** We define a  $K$ -ary expansion score by interpreting the sequence of digits beginning with  $d_i$  as a base  $K$  expansion of a real number between 0 and 1 in the following way

$$\phi_i(d_1 \cdots d_M) = \frac{K - 1 - d_i}{K} + \sum_{j=1}^{\infty} \frac{d_{i+j}}{K^{j+1}}, \quad \forall i : 0 \leq i \leq M \quad (45)$$

where we define  $d_0 = 0$  and  $d_n = K - 1$  for  $n > M$ . Thus for example, the node  $x = 0010100$  for  $K = 2, M = 7$  would have  $\phi_3(x) = 0.00100\bar{1}_2 = 0.00101_2 = 5/32$ . The idea is that the position in the node address describing the smallest real value will determine the level of the node. The leading digit is complemented base  $K$  so that the starting point of this lowest value is clearly marked; without this complementation the resulting tile would have multiple broken edges whenever a very small number (long sequence of 0's) initiates the node address sequence  $d_1 \cdots d_M$ . The boundary conditions are similarly chosen to minimize broken edges.

With this score function, we then define a tile by defining a stratification of nodes through

$$\sigma(x) = i : \phi_i(x) \leq \phi_j(x) \quad \forall j \neq i, 0 \leq j \leq M. \quad (46)$$

There are (limited) circumstances in which two positions  $i$  can have the same value of  $\phi_i(x)$  for a fixed  $x$ , leading to “ties” where multiple values of  $i$  satisfy (46). Such ties only occur for nodes in which the last  $n$  digits are all equal to  $K - 1$  and all previous digits are  $< K - 1$ . In this case we define

$$\sigma((d_1 < K - 1) \cdots (d_{M-n} < K - 1)(d_{M-n+1} = K - 1) \cdots (d_M = K - 1)) = M. \quad (47)$$

With this assignment of levels, each of these nodes has  $K - 1$  children at level  $M - 1$  (those with  $d_M \neq K - 1$ ) and one child at level  $M$  giving a broken edge.

**Example:** In the tile with  $K = 2, M = 6$ , consider the node  $d_1 \cdots d_6 = 010010$ . The smallest score is found at the position where the smallest number appears base 2, with the first bit in the sequence beginning at that point complemented. This is position  $i = 2$ , with  $\phi_2(010010) = 0.00010\bar{1}_2 = 0.00011_2 = 3/32$ . So this node is placed at level  $\sigma(010010) = 2$ . Similarly, each of the child nodes 100100 and 100101 have the smallest  $\phi_i$  at  $i = 1$  so  $\sigma(100100) = \sigma(100101) = 1$ , so edges  $(010010, 100100)$  and  $(010010, 100101)$  are both included in the tile. Note that for the node 100100,  $\phi_1(100100) = 5/64$ , while  $\phi_4(100100) = 1/8$ .

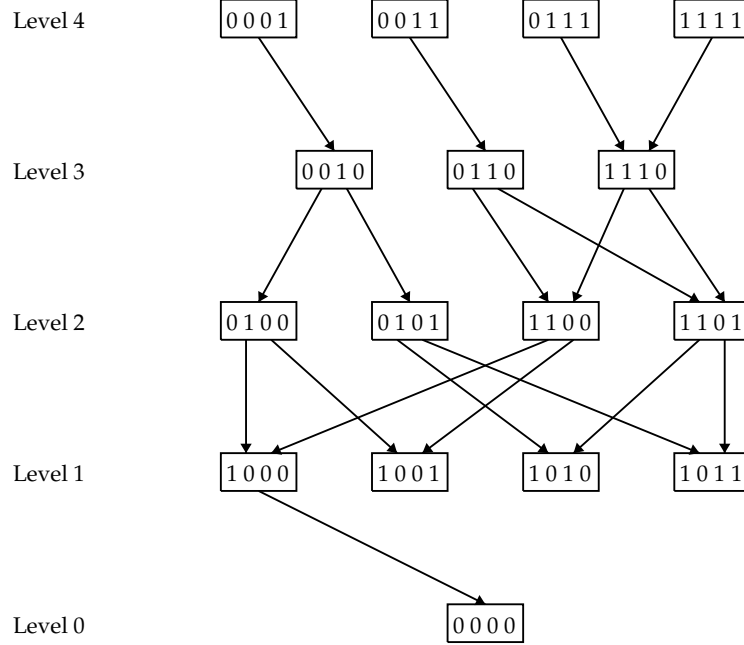


Figure 9: A tile with  $K = 2, M = 4$  based on the 2-ary expansion score stratification. Nodes with the smallest string beginning at position  $i$  (preceded by a 1) are placed on level  $5 - i$ , with boundary conditions and tie-break conditions as discussed in text. Tile is optimal, having 19 internal edges (shown) and 13 external edges (not shown).

We can use this method based on the  $K$ -ary expansion score to construct the full tile for any  $K, M$ . For each node in the tile the level of the stratification is defined through (46), with (47) used to break ties.

**Example:** For the case of  $K = 2, M = 4$ , the  $K$ -ary expansion score stratification gives the tile shown in Figure 9. This tile has 19 internal edges and 13 broken edges. Thus, this algorithm for tile construction yields an optimal tile in this case.

It is straightforward to automate the construction of tiles according to this system. The numbers of internal (broken) edges for tiles with small values of  $K, M$  are listed in the following table.

$K \backslash M$	2	3	4	5	6
2	3* (5)	8* (8)	19* (13)	42 (22)	90 (38)
3	10 (17)	41 (40)	146 (97)	485 (244)	1559 (628)
4	23 (41)	129 (127)	615 (409)	2729 (1367)	11697 (4687)
5	44 (81)	314 (311)	1876 (1249)	10414 (5211)	55794 (22331)

In this table we denote with an asterisk (\*) cases where the tile is known to be optimal. The  $2^5$  tile with 42 internal edges may also be optimal; we have not found a better tile using heuristic search methods. For most of the other small tiles, we have found slightly better

tiles than those constructed using this method with brute force or greedy algorithm searches, or by slight modifications of the score-based stratification algorithm described above; for example as mentioned above for the  $3^4$  tile we have found a tile with 150 internal edges. For larger tiles, we do not know of any general method for generating tiles with significantly more internal edges. As we now discuss, we expect that the asymptotic behavior of these tiles may be the optimal achievable. Note, however, that for larger values of  $K$  the approach to the asymptotic form is slower and the tiles produced by this method are somewhat sub-optimal. In particular, when  $K$  is large and  $M$  is small, many nodes have no digits  $d_i = K - 1$ , which leads to extra broken edges. This might be improved by some heuristic method combining such nodes into sub-tiles which are then attached to increase total edge numbers. But, as  $M \rightarrow \infty$ , for any fixed  $K$  the fraction of such nodes goes to 0 exponentially as  $[(K - 1)/K]^M$ , so this issue does not affect the asymptotics.

Note that in [9] a somewhat similar approach was taken to tiling degree 2 de Bruijn graphs  $\mathcal{B}_2^N$ . In the language of score-based tiles used here, the tilings described in that paper can be defined by assigning a score of  $\phi_i = 0$  when  $d_i d_{i+1} = 01$  and  $\phi_i = 1$  otherwise, with ties broken by choosing  $k$  to be the largest  $i$  with  $\phi_i = 0$ . As described in [9], this gives a tile for  $M = 5$  with  $E_T = 32$  internal edges (compared to 42 from the table above for the system defined here). The tiles described in [9] also do not have good asymptotic behavior, although as mentioned in that paper a generalization of their approach to match longer sequences can improve behavior in a fixed range of  $N$ .

## 7.2 Asymptotics of tiles for local pattern-based scores

We now analyze the asymptotic behavior of the number of broken edges in the  $K$ -ary expansion score-based tiles for fixed  $K$  as  $M \rightarrow \infty$ . We find that these tiles have the same asymptotic form for the number of broken edges as the lower bound (12), multiplied by an overall factor of  $c = 2$ . This asymptotic behavior can be understood from a simple idealized model for the score-based tiles. We give a simple proof of the asymptotics of the number of broken edges in this idealized model, and then describe how the tiles just defined deviate from the idealized model and the consequences of these deviations for the asymptotic form of the number of broken edges. In Section 7.2.3 we numerically analyze the score-based tiles and compare to the theoretical asymptotics.

### 7.2.1 Idealized model

Consider an infinite random walk on the de Bruijn graph  $\mathcal{B}_K^M$  underlying a given tile. Such a random walk will on average traverse each edge an equal number of times, so the fraction of edges traversed which are broken will be equal to the fraction of broken edges on the tile. In an idealized model of the score-based tile system described in the previous subsection, we assume that each node of size  $M$  has scores  $\phi_i$  which are uniformly and independently distributed random numbers between 0 and 1. We assume that the scores are completely

local, so that a node with the score sequence  $\phi_1 \cdots \phi_M$  has children with score sequences  $\phi_2 \cdots \phi_M \phi_{M+1}$  for various values of  $\phi_{M+1}$ . In the idealized model we can associate the random walk on the de Bruijn graph underlying the tile with an infinite sequence of numbers  $x_1 x_2 \cdots$ , each chosen from a uniform random distribution on the set  $[0, 1)$ . The values of  $x_i$  correspond to the sequence of scores  $1 - \phi_i$  on the nodes encountered in the random walk (while for  $\phi_i$  we used the convention that the smallest  $\phi$  determined the level of a node, we reverse this convention for the  $x$ 's to simplify the computations below). In this idealization we ignore correlations between the values of  $\phi_i$  at different points  $i$ , so each  $x_i$  is chosen independently from the uniform distribution. We associate each “node” in the random walk with a subsequence of  $M$  numbers  $S_i = x_{i+1} x_{i+2} \cdots x_{i+M}$ . In this model, there are no ties. An edge  $(x_1 \cdots x_M, x_2 \cdots x_M x_{M+1})$  is broken only if either  $x_1$  or  $x_{M+1}$  is greater than all of  $x_2, \dots, x_M$ . Heuristically, the chances of either occurrence are  $1/M$ , giving a probability of broken edges of  $2/M$ .

To make this computation more precise, for each consecutive subsequence of  $M$  numbers  $S_i = x_{i+1} x_{i+2} \cdots x_{i+M}$  representing a node we assign a level  $k$  where  $x_{i+k}$  is the largest value in the subsequence. We thus have a sequence of values  $k(i)$  defining the positions of the largest  $x$ 's in each subsequence  $S_i$ . If a value  $x_{i+k}$  is the largest  $x$  in subsequence  $S_i$ , and also is the largest  $x$  in  $S_{i+1}$ , then  $k(i+1) = k(i) - 1$ . We say that the sequence has a “broken edge” at position  $i$  when  $k(i+1) \neq k(i) - 1$ . We can now show

**Theorem 4** In this idealized model, as  $M \rightarrow \infty$  the probability of a broken edge asymptotically approaches  $2/M$ .

### Proof

Consider the set of “local maximum”  $x_i$ 's which are maximum values in some subsequence  $S_j$  in which they are contained. There is a broken edge precisely when the maximum  $x$  in one subsequence  $S_i$  is replaced by a new maximum  $x'$  in the subsequence  $S_{i+1}$ . Thus, the frequency of broken edges is the same as the frequency of local maxima. For a given number  $x$  we can determine the probability that it is the largest  $x$  in a window of size precisely  $m$ . For example,  $x_i = x$  is the largest in a window of size precisely 1 when  $x_{i-1} > x, x_{i+1} > x$ . This occurs with probability  $(1-x)^2$ .  $x$  is the largest in a window of size precisely 2 when either  $x_{i-2} > x, x_{i-1} < x, x_{i+1} > x$  or the symmetric condition with  $x > x_{i+1}$  but  $x_{i-1}, x_{i+2} > x$ . The probability that  $x$  is largest in a window of size 2 is therefore  $2x(1-x)^2$ . A similar set of conditions give  $m$  independent cases with probability  $x^{m-1}(1-x)^2$  where  $x$  is the maximum in a window of size  $m$ . Integrating over all possible  $x$ , the probability that a given  $x$  will be largest in a window of precisely size  $m$  is

$$p(m) = \int_0^1 m x^{m-1} (1-x)^2 = m \int_0^1 x^{m-1} - 2x^m + x^{m+1} \quad (48)$$

$$= m \left[ \frac{1}{m} - \frac{2}{m+1} + \frac{1}{m+2} \right] = \frac{2}{(m+1)(m+2)}. \quad (49)$$

the frequency of local maxima is then

$$f = \sum_{m=M}^{\infty} p(m) = \frac{2}{M+1} \quad (50)$$

This proves that in the idealized model the frequency of broken edges is  $2/(M+1) \approx 2/M + \mathcal{O}(1/M^2)$ .

This result in the idealized model suggests that the smallest fraction of broken edges we can achieve in a tile based on local patterns in the node address is asymptotically  $2/M$ . Now let us consider the relevant differences between the idealized model and the  $K$ -ary score-based tiles defined in the previous subsection.

### 7.2.2 Deviations from idealized model

There are three primary ways in which the score-based tiles we have defined deviate from the idealized model. First, there are some situations in which different positions have the same score  $\phi_i = \phi_j$  for a fixed node. Second, the scores are not completely independent. Third, the scores  $\phi_i$  depend on multiple local symbols  $d_k$ , and therefore in particular the score associated with a point  $i$  in a node does not necessarily stay invariant in the children of that node—we have  $\phi_{i-1}(v) \neq \phi_i(u)$  along an edge  $(u, v)$  whenever the final symbol in  $v$  is not  $K-1$ .

Let us treat these deviations from the idealized model in order. First, consider the situation of ties. Generally, ties will increase the frequency of broken edges. For example, consider the variation on the above model where  $x_i$  is chosen randomly from a uniform distribution on the discrete set  $\{k/D, 0 \leq k \leq D\}$ , and ties are broken by choosing  $k(i)$  to be the smallest  $k$  with the maximum value of  $x_{i+k}$  in the set  $S_i$ . Then when  $M \gg D$ , the frequency of broken edges will be  $1/D \gg 1/M$ . (The approach taken in [9] is an extreme example of this; their rule for constructing tiles can be formulated in the language of scores as described above by taking a score which is either  $\phi_i = 0$  if  $d_i d_{i+1} = 01$  or  $\phi_i = 1$  otherwise. This gives a frequency of broken edges of  $1/4$  for large  $M$ .)

As noted above, for the  $K$ -ary score based tiles defined in 7.1, ties only occur for nodes with  $d_1, \dots, d_{n-1} < K-1, d_n = \dots d_M = K-1$ . The total number of nodes where such ties occur is

$$N_{\text{ties}} = 1 + (K-1)^1 + (K-1)^2 + \dots + (K-1)^{M-1} = \frac{(K-1)^M - 1}{K-2}. \quad (51)$$

The fraction of nodes where ties happen therefore decreases exponentially in  $M$  as  $[(K-1)/K]^M \ll 1/M$  when  $M \rightarrow \infty$ . The consequences of ties are therefore negligible in the asymptotic behavior of the number of broken edges.

Next, let us consider the other two issues: the lack of independence between nearby scores and the non-invariance of scores when edges are followed. Each of these effects arises from dependence of the score  $\phi_i$  on all the symbols  $d_{i+k}$  for any  $k > 0$ . From the form of (45) we



see that the effect of  $d_{i+k}$  on the score  $\phi_i$  is suppressed exponentially as  $1/K^k$ . For a node in a tile of size  $M$ , the average spacing between the scores of the  $M$  different positions  $i$  is  $1/M$ . Thus, the impact of  $d_{i+k}$  becomes negligible compared to this spacing when  $k \gg \log_K M$ . Another way to see this is to note that the smallest  $\phi$  will arise from the longest sequence of 0's in the node which is preceded by the symbol  $K - 1$ . As  $M \rightarrow \infty$ , the average number of sequences of  $n$  zeros in a random node address goes as  $M/K^n$ . Thus, we expect  $\mathcal{O}(1)$  sequences of  $\log_K M$  0's but  $\mathcal{O}(1/M^n)$  sequences of  $n + 1$  times this many 0's. So again, the number of relevant digits in determining the minimum  $\phi$  is of order  $\mathcal{O}(\log_K M)$ . The relevant correlation distance between  $\phi$ 's is thus also  $\log_K M$ . Similarly, when a new digit sequence giving the lowest  $\phi$  is being shifted in from the right, asymptotically of order  $\mathcal{O}(\log_K M)$  digits must be shifted in to realize the smaller value of  $\phi$ .

The upshot of this analysis is that there are corrections to the asymptotic form of the idealized model of order  $(\log_K M)/M$  when we consider the score-based tiles described above. The most important of these effects is the delay by  $\log_K M$  digits in shifting in a new score  $\phi_i$ . When  $M - i < \log_K M$ ,  $\phi_i$  will be higher than the appropriately shifted value in any of the descendants of a given node due to the boundary condition  $d_n = K - 1$  for  $n > M$  described below (45). This means that the effective size of the window associated with the tile is really  $M - \mathcal{O}(\log_K M)$ , which contributes a correction term of order  $(\log_K M)/M$  to the asymptotic form of the fraction of broken edges  $2/M$  for the idealized model demonstrated in Theorem 4.

Because all of these effects are either exponentially suppressed in  $M$  or suppressed by a factor of  $(\log_K M)/M$  relative to the leading  $2/M$ , none of these effects will modify the asymptotic form of the number of broken edges for the score-based tiles described above. We now verify this analysis with numerical computation.

### 7.2.3 Numerical verification of asymptotics

To verify the asymptotic analysis performed in the previous subsection, and the validity of the approximations made in going from the score-based tiling system to the idealized model, we have done a numerical analysis of the number of broken edges for tiles with small  $K$  and reasonably large  $M$ . This data corresponds closely with the theoretical analysis just performed.

In Figure 10 we have graphed the logarithm of the number of broken edges in the 2-ary expansion score-based tiles for  $K = 2, M \leq 20$ . The number of broken edges in these tiles matches very closely to the theoretically estimated fraction of  $2/M$ , and lies above the asymptotic form of the minimum fraction  $1/M$ . Note that for  $M > 11$ , the number of broken edges exceeds the fraction  $2/M$  of the total number of edges. This is compatible, however, with the expected  $(\log_K M)/M$  form of the corrections to the asymptotic form computed above. For  $M > 11$  the computed number of broken edges lies between the asymptotic form  $2/M$  and the log-corrected asymptotic form  $2/(M + 1 - (\log_K M)/2)$ .

For tiles with  $K = 3$  we have a similar close match to the asymptotic fraction  $2/M$  of

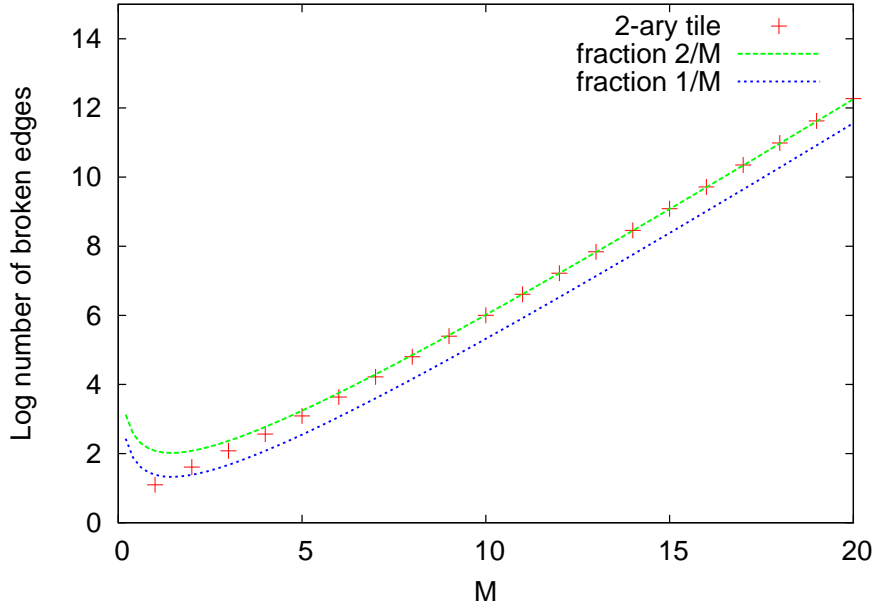


Figure 10: (Logarithm of) number of broken edges for 2-ary expansion score based tile, compared to theoretical asymptotic fraction  $2/M$  and asymptotic lower bound  $1/M$ .

broken edges, as depicted in Figure 11.

To summarize, from our analysis of an idealized model, we expect that a fraction of  $2/M$  broken edges is asymptotically the optimum which can be achieved based on a local pattern for level determination. We have constructed a family of tiles which realize this asymptotic form, and are thus asymptotically optimal for local pattern-based tile structures. This asymptotic result differs from the lower bound computed in Section 3 by a factor of 2. We suspect that it is not possible to find tilings with an asymptotic fraction of less than  $2/M$  broken edges, based on our asymptotic analysis and brute force and greedy algorithm analysis of small tiles, but we have not proven this conclusively. Any mechanism for constructing tiles with an asymptotic fraction of  $c/M$  broken edges with  $c < 2$  would require a means of determining the level of each node based on global properties of the  $dd$  node address, like some kind of global hashing function, rather than a determination of level based on local patterns such as we have considered here.

## 8 Application to supercomputers

The results of this paper give not only a theoretical understanding of how de Bruijn and Kautz graphs can be decomposed into isomorphic subgraph tiles, but also a concrete approach to constructing such tilings. Computer systems containing thousands of individual processing elements need efficient communication networks to minimize overhead in passing data between the processors. Because of their high degree of connectivity, de Bruijn and

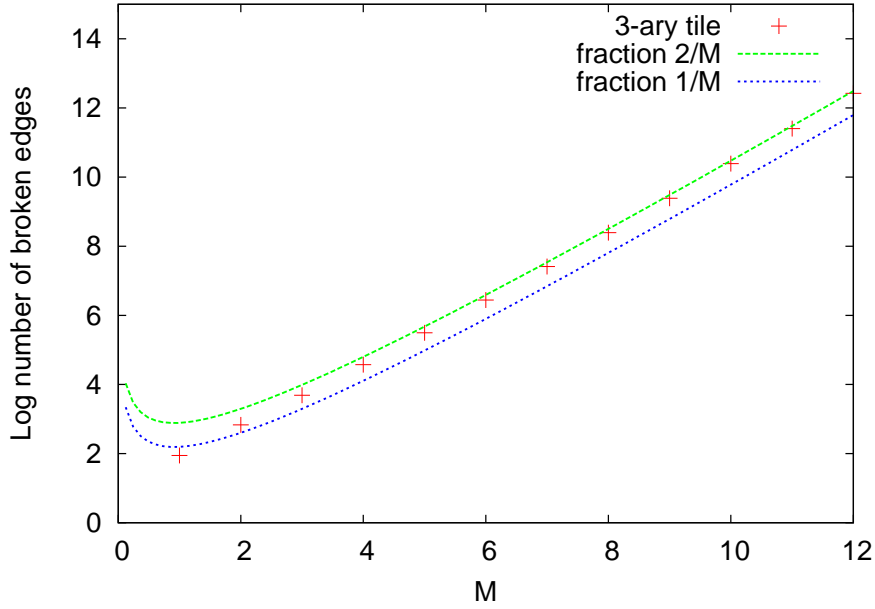


Figure 11: (Logarithm of) number of broken edges for 3-ary expansion score based tile, compared to theoretical asymptotic fraction  $2/M$  and asymptotic lower bound  $1/M$ .

Kautz graphs are very well suited to such large scale processing networks. The practical problem of wiring together many processors in such a network is substantially simplified by the approach of combining multiple processing units into tiles with isomorphic wiring, and then combining the tiles as we have described in this paper.

In principle, the methods described in this paper can be used to design wiring systems for computing systems at a range of scales. For any given size of system, the tradeoff between communication and processing power will affect the choice of degree  $K$ , and practical design considerations will affect the choice of tile size  $K^M$ . The results described here should be useful in determining the complexity of wiring needed for any such system design. In particular, the lower bound on edges connecting tiles derived in Section 3 gives an absolute lower limit on the complexity of wiring necessary for such a system. The explicit tile constructions given here give concrete examples of wiring patterns which can be used for such systems.

The feasibility of a practical implementation of the tiling methods developed in this paper is demonstrated by the Sicortex, Inc. family of cluster computer systems [12]. The internal communication network in these systems is based on a degree  $K = 3$  Kautz digraph. Circuit boards contain processor nodes connected in a stratified subgraph of a degree 3, diameter 3 de Bruijn digraph, so that a full Kautz graph of any desired size can be wired by connecting identical boards as described in Theorem 3, taking advantage of the parallel routing property.

## 9 Conclusions

In this paper we have developed a systematic approach to partitioning de Bruijn and Kautz graphs into isomorphic subgraph “tiles” connected by a minimal number of additional edges. These results utilize the common mathematical structure underlying de Bruijn and Kautz graphs, and shed light on the structure of these graphs and their generalizations. The tilings we have constructed here have practical application to the construction of massively parallel computer systems.

We have given necessary and sufficient conditions for constructing efficient tilings, we have characterized optimal tiles in terms of de Bruijn graphs of the size of the desired tile, and we have constructed an asymptotically optimal class of tilings. We have not, however, found a general method for explicitly constructing provably optimal tiles of arbitrary size, nor have we found a general formula for the number of internal edges achievable by an optimal tile of arbitrary size. We leave these open problems for future work.

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